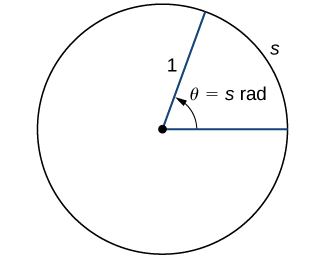
# Section 1.3: Trigonometric Functions

In this section, we define the six basic trigonometric functions and look at some of the main identities involving these functions.

## Radian Measure

To use trigonometric functions, we first must understand how to measure angles. Although we can use both radians and degrees, radians are a more natural measurement because they are related directly to the unit circle, a circle with radius 1.

The radian measure of an angle is the arc length of the associated arc on the unit circle.



Since the angle of corresponds to the circumference of a circle, or an arc of length . We conclude that an angle with a degree measure of has a radian measure of . Similarly, we see that is equivalent to radians.

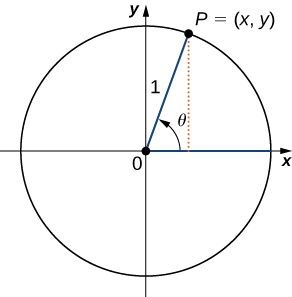
Examples

1. Express using radians.
2. Express radians using degrees.

## The Six Basic Trigonometric Functions

Trigonometric functions allow us to use angles measures, in radians or degrees, to find the coordinates of a point on any circle – not only on a unit circle – or to find an angle given a point on a circle. They also define the relationship among the sides and angles of a triangle.

Let be a point on the unit circle centered at the origin . Let be an angle with an initial side along the positive -axis and a terminal side given by the line segment .

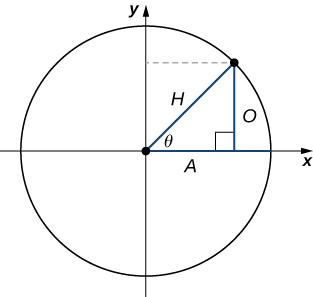


The trigonometric functions are then defined as

If , and are undefined. If , and are undefined.

In addition, the ratios of the side lengths of a right triangle can be expressed in terms of the trigonometric functions evaluated at either of the acute angles of the triangle.

Let be the length of the adjacent leg, be the length of the opposite leg, and be the length of the hypotenuse. By inscribing the triangle into a circle with radius , as shown below



We see that , and satisfy the following relationships with :

Examples

1. Evaluate each of the following expressions:
2. is a point on the unit circle. Find the exact missing coordinate value of each point and find the values of the six trigonometric functions for the angle with a terminal side that passes through point . Rationalize denominators.
3. A wooden ramp is to be built with one end on the ground and the other end at the top of a short staircase. If the top of the staircase is 4 ft from the ground and the angle between the ground and the ramp is to be , how long does the ramp need to be?

## Trigonometric Identities

A trigonometric identity is an equation involving trigonometric functions that is true for all angles for which the functions are defined. We can use the identities to help us solve or simplify equations.

The main trigonometric identities are:

**Reciprocal Identities**

**Pythagorean Identities**

**Addition and Subtraction Formulas**

**Double-Angle Formulas**

Examples

1. For each of the following equations, use a trigonometric identity to find all solutions.
2. Prove the trigonometric identity .

## Graphs and Periods of the Trigonometric Functions

We have seen that as we travel around the unit circle, the values of the trigonometric functions repeat. We can see this pattern in the graphs of the functions.

An image of six graphs. Each graph has an x axis that runs from -2 pi to 2 pi and a y axis that runs from -2 to 2. The first graph is of the function “f(x) = sin(x)”, which is a curved wave function. The graph of the function starts at the point (-2 pi, 0) and increases until the point (-((3 pi)/2), 1). After this point, the function decreases until the point (-(pi/2), -1). After this point, the function increases until the point ((pi/2), 1). After this point, the function decreases until the point (((3 pi)/2), -1). After this point, the function begins to increase again. The x intercepts shown on the graph are at the points (-2 pi, 0), (-pi, 0), (0, 0), (pi, 0), and (2 pi, 0). The y intercept is at the origin. The second graph is of the function “f(x) = cos(x)”, which is a curved wave function. The graph of the function starts at the point (-2 pi, 1) and decreases until the point (-pi, -1). After this point, the function increases until the point (0, 1). After this point, the function decreases until the point (pi, -1). After this point, the function increases again. The x intercepts shown on the graph are at the points (-((3 pi)/2), 0), (-(pi/2), 0), ((pi/2), 0), and (((3 pi)/2), 0). The y intercept is at the point (0, 1). The graph of cos(x) is the same as the graph of sin(x), except it is shifted to the left by a distance of (pi/2). On the next four graphs there are dotted vertical lines which are not a part of the function, but act as boundaries for the function, boundaries the function will never touch. They are known as vertical asymptotes. There are infinite vertical asymptotes for all of these functions, but these graphs only show a few. The third graph is of the function “f(x) = csc(x)”. The vertical asymptotes for “f(x) = csc(x)” on this graph occur at “x = -2 pi”, “x = -pi”, “x = 0”, “x = pi”, and “x = 2 pi”. Between the “x = -2 pi” and “x = -pi” asymptotes, the function looks like an upward facing “U”, with a minimum at the point (-((3 pi)/2), 1). Between the “x = -pi” and “x = 0” asymptotes, the function looks like an downward facing “U”, with a maximum at the point (-(pi/2), -1). Between the “x = 0” and “x = pi” asymptotes, the function looks like an upward facing “U”, with a minimum at the point ((pi/2), 1). Between the “x = pi” and “x = 2 pi” asymptotes, the function looks like an downward facing “U”, with a maximum at the point (((3 pi)/2), -1). The fourth graph is of the function “f(x) = sec(x)”. The vertical asymptotes for this function on this graph are at “x = -((3 pi)/2)”, “x = -(pi/2)”, “x = (pi/2)”, and “x = ((3 pi)/2)”. Between the “x = -((3 pi)/2)” and “x = -(pi/2)” asymptotes, the function looks like an downward facing “U”, with a maximum at the point (-pi, -1). Between the “x = -(pi/2)” and “x = (pi/2)” asymptotes, the function looks like an upward facing “U”, with a minimum at the point (0, 1). Between the “x = (pi/2)” and “x = (3pi/2)” asymptotes, the function looks like an downward facing “U”, with a maximum at the point (pi, -1). The graph of sec(x) is the same as the graph of csc(x), except it is shifted to the left by a distance of (pi/2). The fifth graph is of the function “f(x) = tan(x)”. The vertical asymptotes of this function on this graph occur at “x = -((3 pi)/2)”, “x = -(pi/2)”, “x = (pi/2)”, and “x = ((3 pi)/2)”. In between all of the vertical asymptotes, the function is always increasing but it never touches the asymptotes. The x intercepts on this graph occur at the points (-2 pi, 0), (-pi, 0), (0, 0), (pi, 0), and (2 pi, 0). The y intercept is at the origin. The sixth graph is of the function “f(x) = cot(x)”. The vertical asymptotes of this function on this graph occur at “x = -2 pi”, “x = -pi”, “x = 0”, “x = pi”, and “x = 2 pi”. In between all of the vertical asymptotes, the function is always decreasing but it never touches the asymptotes. The x intercepts on this graph occur at the points (-((3 pi)/2), 0), (-(pi/2), 0), ((pi/2), 0), and (((3 pi)/2), 0) and there is no y intercept.

Trigonometric functions are **periodic functions**. The **period** of a function is defined to be the smallest positive value such that for all values in the domain of .

The sine, cosine, secant, and cosecant functions have a period of . Since the tangent and cotangent functions repeat on an interval of length , their period is .

Just as with algebraic functions, we can apply transformations to trigonometric functions.

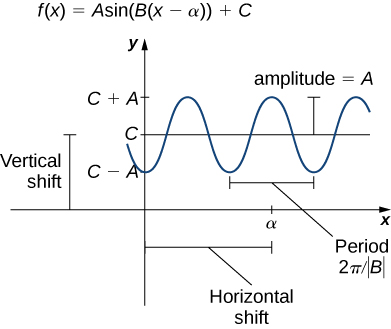
Consider the following function:

The factor results in a vertical stretch by a factor of . We say is the “amplitude of .”

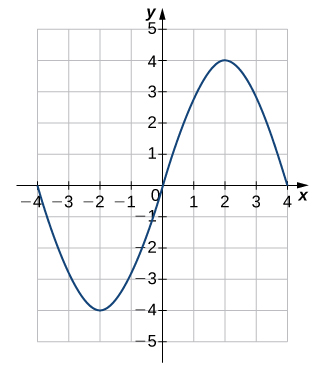
The factor changes the period.

The constant causes a horizontal or phase shift.

The constant causes a vertical shift.



Examples

1. For the function, , find the amplitude, the period, and the phase shift.
2. Sketch a graph of .
3. For each of the following graphs, write the equation of the graph in the form or , where .
   1. 
   2. 